Algebra + Homotopy = Operad. (Bruno. Vallette)

Abelian category $\mathcal{A}$, $(A, d_A), (H, d_H) \in Ch(\mathcal{A})$

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

h degree 1. $\quad Id_A - ip = d_A h + h d_A$.

If i. quasi-isomorphism, then homotopy retract.

A. additional structure, then what about H?

e.g. take $H = H(A)$. $d_H = 0$.

A. A_∞-algebra, then $H(A)$ minimal model
$/L_\infty, E_\infty$

Operads $\quad$ group representation. $\quad G \longrightarrow End(V)$

$$(V, g_1, g_2, \cdots)$$
$$V \to V$$

$$V \otimes V \to V. \qquad V \otimes V \otimes V \longrightarrow V$$

Fin : category $\{[0], [1], [2], \cdots\}$ where morphisms are bijections.

$[n] = \{1, 2, \ldots n\}$. $\quad [0] = \emptyset$.

$$Hom([n], [n]) = S_n$$

Ord : category $\{[0], [1], [2], \cdots\}$. where morphisms are order-preserving bijections

$Hom([n], [n]) = \{id\}$.

Ord is discrete category.

Let $\mathcal{C}$ be sym. monoidal category

A (ns) non-symmetric sequence in $\mathcal{C}$ is functor Ord $\to \mathcal{C}$.

A (symmetric) sequence in $\mathcal{C}$ is functor Fin $\to \mathcal{C}$.

$\mathcal{X}$ is sequence. $\mathcal{X}(0), \mathcal{X}(1), \dots \in \mathcal{C}$.

symmetric case: action of S_n on $\mathcal{X}(n)$

Composition of sequences: (assume $\mathcal{C}$ has finite coproducts)

$$(\mathcal{X} \circ_{ns} \mathcal{Y})(n) = \bigoplus_{\lambda \vdash n} \left(\mathcal{X}(r) \otimes \mathcal{Y}(\lambda_1) \otimes \cdots \otimes \mathcal{Y}(\lambda_r)\right)$$

λ iterates through partitions of n. $\lambda_1 + \dots + \lambda_r = n$

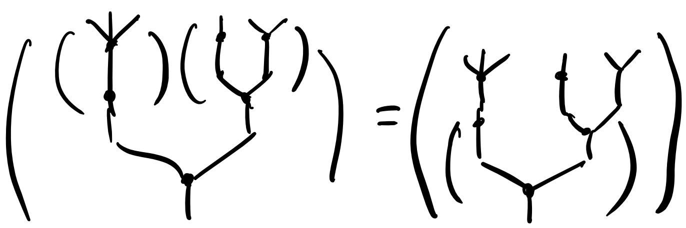
$$(\mathcal{X} \circ_{sym} \mathcal{Y})(n) = \bigoplus_{\lambda \vdash n} \left(\mathcal{X}(r) \otimes_{S_r} \underbrace{\bigoplus_{\sigma \in S_r} \left(\mathcal{Y}(\lambda_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathcal{Y}(\lambda_{\sigma^{-1}(r)})\right)}_{\text{has canonical } S_r \text{ action}}\right)$$

ns operad is an ns sequence $\mathcal{O}$ with composition law $\mu : \mathcal{O} \circ_{ns} \mathcal{O} \to \mathcal{O}$

(sym) operad is a (sym) sequence with composition law $\mu : \mathcal{O} \circ_{sym} \mathcal{O} \to \mathcal{O}$

(natural transformation)

$$\mu_n : \mathcal{O}(r) \otimes \mathcal{O}(\lambda_1) \otimes \cdots \otimes \mathcal{O}(\lambda_r) \longrightarrow \underline{\mathcal{O}(n)}$$



with unit $1 \in \mathcal{O}(1)$ $\quad \mu(\underset{\mathcal{O}(r)}{f}, 1, 1, \dots 1) = f = \mu(1, f)$

Alternative definition of operad. (partial composition)

$$\circ : \mathcal{O}(r) \otimes \mathcal{O}(s) \longrightarrow \mathcal{O}(s+r-1)$$

$$\mu_n : (f_1, \dots f_{r+1}) \longmapsto ((f_1 \circ f_2) \circ f_3) \cdots \circ f_{r+1}$$

$\mathcal{D}$ is $\mathcal{C}$-enriched category. $V \in \mathcal{D}$.

$\mathrm{Hom}(V)$ is an operad (ns/sym)

$$\mathrm{Hom}(V)_n = \mathrm{Hom}_{\mathcal{D}}(V^{\otimes n}, V)$$

$$\sigma \in S_n : f \longmapsto \left((x_1, \dots x_n) \mapsto f(x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(n)})\right)$$

let $\mathcal{O}$ be an operad, then an $\mathcal{O}$-algebra is a morphism

$$A : \mathcal{O} \longrightarrow \mathrm{Hom}(V)$$

$A_n : \mathcal{O}(n)\otimes V^{\otimes n} \longrightarrow V \qquad (n=0:\ \mathcal{O}(n)\longrightarrow V)$

Morphism of operads. Let P, Q be operads.

A morphism of operads $f: P\to Q$ is a natural transformation

commuting with μ:

$$\begin{tikzcd} P\circ P \arrow[r, "\mu"] \arrow[d, "f"] & P \arrow[d, "f"] \\ Q\circ Q \arrow[r, "\mu"] & Q \end{tikzcd}$$

$\overset{ns}{As}$ / $\overset{sym}{Ass}$ non-unital associative $\mathbb{K}$-algebra $\ell = \mathbb{1} = \mathbb{K}$-vector space

$As(2) \ni Y = \times$ $\qquad$ $\underline{Y}$ $=$ $\underline{Y}$ $\in As(3)$ $\quad \cdots$

$As(n) = \mathbb{K}\, \overset{n}{Y} \quad (n\ge 1) \qquad As(0) = 0$

μ = trivial

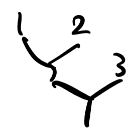
As-algebra V. $\quad a,b,c\in V$.

$Y(Y(a,b),c) = \Psi(a,b,c) = Y(a,Y(b,c))$

sym. operad. $\quad Y(a,b)\neq Y(b,a) \qquad \underbrace{a,b\mapsto a\cdot b}_{e\in S_2} \quad \underbrace{a,b\mapsto b\cdot a}_{(12)\in S_2}$

$n\ge 1$. $Ass(n) = \mathbb{K}S_n$

→ operation are labelled trees.



$\overset{1\,2}{Y}(a,b) = \overset{2\,1}{Y}(b,a)$

$\overset{(sym)}{Com}$ commutative assoc. $\mathbb{K}$-algebra.

$\overset{1\,2}{Y} = \overset{2\,1}{Y}$. $\qquad Com(n) = \mathbb{K}\,\overset{1\,2\,\cdots\,n}{Y}$

$\sigma\in S_n. \quad \sigma : Com(n) \longrightarrow Com(n). \quad \sigma = id$

$\overset{(sym)}{Lie}$. Lie algebra. $\quad Y \qquad$ [MR96]

Unital trick. uAs: ns operad encoding unital assoc. $\mathbb{K}$-algebra.

$uAs(n) = As(n)$ for $n\ge 1$. $\qquad uAs(0) = \mathbb{K}\cdot 1$

A. uAs-algebra: $uAs \to \mathrm{Hom}(V)$.

$A_0(1) = 1_V$.

Likewise. uCom.

$\mathcal{C}$ = Set. $\mathcal{D}$ = Set (ns) Monoid operad. $Mon(n) = \{ \ \}$. $n \geq 0$.

Mon-algebra = monoids.

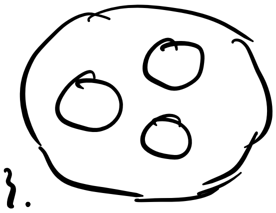
Free: $(Set, \times) \longrightarrow (\mathbb{K}\text{-Vect}, \otimes)$

Free $\circ$ Mon = uAs. (Likewise. $\overline{Mon}$ where $\overline{Mon}(0) = \emptyset$

then Free $\circ$ $\overline{Mon}$ = As)

Topological Operad. $\mathcal{C} = (Top, \times)$

e.g. Little cubes. elements in $D^k(n)$. are

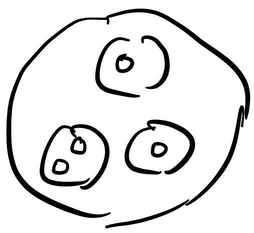
 $\in D^2(3)$

collections $\{ f_i : S^{k-1} \to D^k : i \in \{1, \dots, n\} \}$.

f_i non-intersecting.

$(D^k(n) \in Top.$ as $\subseteq \prod_n Hom_{Top}(S^{k-1}, D^k))$

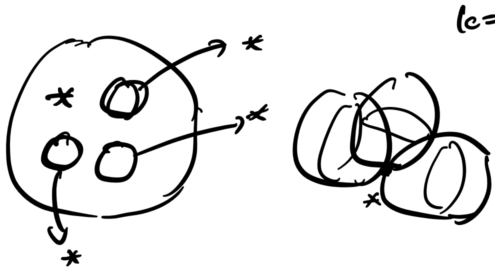
μ_n:



k-fold loop space. $\Omega^k(Y) = \{ (S^k, *) \to (Y, *) \}$

$\hookrightarrow$ as D^k-algebra.

k=2



Recognition Principle (BV72) connected $X \in Top$.

X can be equipped with D^k-algebra iff $X \sim \Omega^k(Y)$

[Coh 76]. $H_\bullet : (Top, \times) \to (gr\text{-}Mod, \otimes)$

$H_\bullet \circ D^2 \cong Ger$ Gerstenhaber operad. (assoc. algebra with Lie bracket)

Extensions to operad.

Properad. $Fin \times Fin \to \mathcal{C}$ $\quad O(m,n)$

e.g. $R(m,n)$ set of Riemann surfaces. m input hole, n. output hole.

[Seg 04]. conformal field theory = R-algebra.

PROP. category of cobordisms form a PROP.

cobordism-algebra = TQFT [Ati 88]

Free operads.

$$\text{Operads} \underset{\text{free}}{\overset{\text{forgetful}}{\rightleftarrows}} [\text{Fin}, \mathcal{C}]$$

$T: [\text{Fin}, \mathcal{C}] \longrightarrow \text{Operads}.$

$[\text{Ord}, \mathcal{C}] \longrightarrow \text{nsOperads}.$

equivalently.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & T(X) \arrow[d, dashed, "\exists!\text{ operad morphism.}"] \\ {[\text{Fin}, \mathcal{C}]} & \mathcal{P} \end{tikzcd}$$

$X \in [\text{Fin}, \mathcal{C}]$, $\mathcal{P} \leftarrow$ operad

Let PT_n denote all planar trees with n leaves.

$t \in PT_n \quad t = (V, E)$

$t(X) = \{ f: V \to X$ s.t. $v \in V$, $\underbrace{d(v)}_{\text{no. of inputs.}} = n \Rightarrow f(v) \in X(n) \}$.

$\Rightarrow T(X)(n) = \bigoplus_{t \in PT_n} t(X).$

If $\mathcal{C} = dg \ldots$ operad $(\mathcal{O}, d)$ $\quad d: \mathcal{O}(n) \to \mathcal{O}(n)$

d derivation : $d(\mu(\overset{\mathcal{O}(k)}{\overset{\in}{x}}, y_1, \ldots y_k)) = \mu(d(x), y_1, \ldots y_k)$
$\qquad + \sum_{i=1}^{k} \pm \mu(x, y_1, \ldots d(y_i), \ldots y_k)$

quasi-free operad: $(\mathcal{O}, d)$ quasi-free = $\mathcal{O}$ free as graded module operad.

quasi-free resolution: $(P, d_P) \longrightarrow (\mathcal{O}, d_\mathcal{O})$

(P, d_P) quasi-free, $\longrightarrow$ quasi-isomorphism $H(P(n), d_P) \cong H(\mathcal{O}(n), d_\mathcal{O})$

$\hookrightarrow$ Koszul duality: A_∞ is quasi-free reso. of As.

E_∞ is ----- of Com

L_∞ is ----- of Lie

Quadratic data and rewriting test.

$\mathcal{P}$ is operad. $I \subseteq P$ is ideal iff $\{$ closed under S_n action closed under composition

$x, y_1, \cdots y_k$ at least one in $I \Rightarrow \mu(x, y_1, \cdots y_k) \in I$.

$\Rightarrow P/I$. also operad.

quadratic data $(E; R)$ s.t. $R \subset \underline{T(E)^{(2)}} = \bigoplus_{\substack{t \in PT^{(n)} \\ |V_t| = 2.}} t(X)$

$\Rightarrow$ quadratic operad $\mathcal{P}(E; R) = T(E)/\underline{\langle R \rangle}$.

e.g. $As. \cong \mathcal{P}(Y ; Y - Y)$

$Com \cong \mathcal{P}(Y ; Y - Y, Y - Y)$

implies $Y = Y$

$Lie \cong \mathcal{P}(Y ; Y - Y - Y)$

$Y = -Y$

Why units are bad. uAs ?

not quadratic. but quadratic linear

$uAs = \mathcal{P}(\underbrace{\iota, Y} ; Y - Y, Y - 1, Y - 1)$

$T(\cdots) \ni Y \neq id \quad Y - 1 \xleftarrow{id} , \ Y - 1$?

Rewriting method : test whether a quadratic data is Koszul $(E; R)$

Assume E is binary. $E(n) = 0$ for $n \neq 2$

ordered basis of E: $x_1 < \cdots < x_l$

Consider left comp. $x_i \circ_1 x_j$; $Y \to x_j,\ Y \to x_i$ right comp. $x_i \circ_2 x_j$: $Y \leftarrow x_j,\ Y \leftarrow x_i$

$R \ni r = \lambda x_i \circ_a x_j - \sum_i \lambda^* x_{r_i} \circ_{b_i} x_{s_i} \quad (\lambda \neq 0)$

$x_i \circ_a x_j$ maximum w.r.t. $x_i \circ_1 x_j > x_p \circ_2 x_q$

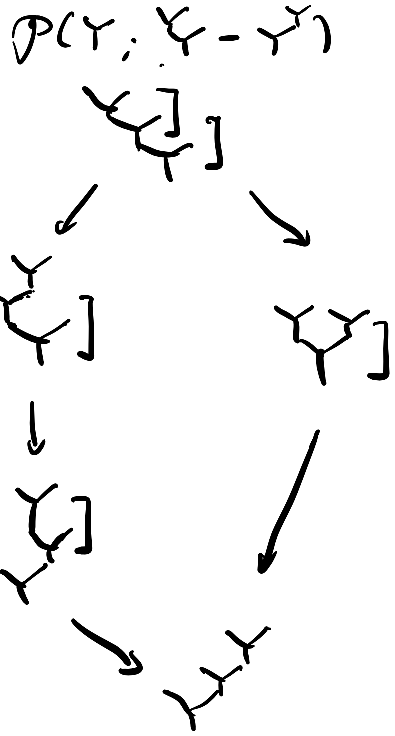
$x_i \circ_a x_j > x_p \circ_a x_q$ when $i > p$ or $i = p$ & $j > q$.

$r \Rightarrow x_i \circ_a x_j \longmapsto \sum_i \lambda^*/\lambda \, x_{r_i} \circ_{b_i} x_{s_i}$

Critical monomial $\in T(E)^{(3)}$ s.t. both sub-trees with 2 vertices are leading.

Rewriting test: if 2 rewrites of any critical monomial commute

$\Rightarrow \mathcal{P}(E; R)$ is Koszul.

e.g. $As =$ 

sym. operad. : order labelled tree. Vallette. p.32

<u>Koszul duality</u> $(E;R)$ quadratic data. then

$$P_\infty = (T(s^{-1}\bar{\mathcal{P}}^{¡}), d_2) \longrightarrow \mathcal{P}(E;R)$$

$(E;R)$ Koszul $\Rightarrow$ morphism is quasi-isomorphism.

$\mathcal{P}(E;R)$ Koszul. $h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$ homotopy retract.

$\Rightarrow$ P_∞-structure on A induce P_∞-structure on H.

s.t. i is P_∞-quasi-isomorphism.